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Distance functions and statistics

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Abstract

This paper proves that the Riemannian distance function is maximal in the class of distance functions associated with the Riemannian metric tensor.

Secondly, it is proven that there exists a unique minimum of

$$L(v) = \sum_{i=1}^{m} d(p_i, \gamma_v(t_i))^2, \quad v \in TM$$

on a complete Riemannian surface (M, g) with small curvature, small curvature change and injectivity radius $+\infty$. Here $p_i \in M$ and γ_v is the maximal geodesic with initial velocity v and $0 < t_1 < \cdots < t_m$.

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1. Distance functions and metric tensors

Let (M, g) denote a smooth, connected Riemannian manifold. Furthermore, let

$$d_g : M \times M \to \mathbb{R}$$

denote the Riemannian distance function of g. For small enough v, we have

$$d_g^2(\pi(v), \exp(v)) = \langle v, v \rangle. \tag{1.1}$$

Let $p = \pi(v)$ and take a chart (U, ϕ) around p. $(\tilde{U}, \tilde{\phi})$ denotes the tangent bundle chart induced by (U, ϕ). Define

$$d_{g,\phi} = d_g^2 \circ (\phi^{-1} \times \phi^{-1}), \qquad \tilde{\phi}(v) = (x, h), \qquad \exp^{\phi}(x, h) = \phi \circ \exp \circ \tilde{\phi}^{-1}(x, h).$$

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Equality (1.1) reads in our local coordinates

$$d_{g,\phi}(x, \exp^{\phi}(x, h)) = \sum_{i,j} g_{ij}(x) h^i h^j.$$

Now

$$\frac{\partial^2}{\partial h_a \partial h_b} \sum_{i,j} g_{ij} h^i h^j_{|h=0} = 2 g_{ab}(x),$$

while

$$\frac{\partial^2}{\partial h_b \partial h_a} d_{g,\phi}(x, \exp^{\phi}(x, h))_{|h=0} = \frac{\partial}{\partial h_b} \left(\frac{\partial d_{g,\phi}}{\partial z_l}(x, \exp^{\phi}(x, h)) \frac{\partial \exp^{\phi}_l}{\partial h_a}(x, h) \right) \Bigg|_{h=0}$$

$$= \frac{\partial^2 d_{g,\phi}}{\partial z_l \partial z_m}(x, \exp^{\phi}(x, h)) \frac{\partial \exp^{\phi}_l}{\partial h_a}(x, h) \frac{\partial \exp^{\phi}_m}{\partial h_b}(x, h)_{|h=0} = \frac{\partial^2 d_{g,\phi}}{\partial z_l \partial z_m}(x, x) \delta^l_a \delta^m_b$$

$$= \frac{\partial^2 d_{g,\phi}}{\partial z_a \partial z_b}(x, x).$$

Hence

$$\frac{\partial^2 d_{g,\phi}}{\partial z_a \partial z_b}(x, x) = 2 g_{ab}(x).$$

We have used that

$$\frac{\partial d_{g,\phi}}{\partial z_l}(x, x) = 0,$$

since $d_{g,\phi} \geq 0$ and $d_{g,\phi}(x, x) = 0$. Also

$$\frac{\partial \exp^{\phi}_l}{\partial h_a} = \delta^l_a,$$

since

$$d \exp_p(0) = \mathrm{id}.$$

We see that a Riemannian distance function $d = d_g$ satisfies

1. $d(p, p) = 0\, \forall p \in M$.
2. $d(p, q) = d(q, p)\, \forall p, q \in M$.
3. $d(p, r) \leq d(p, q) + d(q, r)\, \forall p, q, r \in M$.
4. $d^2 \colon M \times M \to \mathbb{R}$ is C^∞ on an open neighbourhood of the diagonal $\Delta \subset M \times M$ and

$$\left\{ \frac{\partial^2 d_\phi}{\partial z_l \partial z_m}(\phi(p), \phi(p)) \right\}_{l,m}$$

is positive definite $\forall p \in M$ and every chart (U, ϕ) around p.

A function $d\colon M\times M\to\mathbb{R}$ satisfying (1)–(4) above is called a distance function and is said to satisfy the condition R^∞.

So a Riemannian metric tensor gives rise to a distance function d_g satisfying the condition R^∞.

Take $p\in M$ and a chart (U,ϕ) around p. Define

$$g_{kl}(p)=\frac{1}{2}\frac{\partial^2 d_\phi}{\partial z_k\partial z_l}(\phi(p),\phi(p)).$$

This is a coordinate invariant definition of

$$g(p), \tag{1.2}$$

where g is a smooth Riemannian metric tensor.

We shall agree to say that a distance function d satisfying the condition R^∞ is associated with the metric g provided

$$g_{kl}(p)=\frac{1}{2}\frac{\partial^2 d_\phi}{\partial z_k\partial z_l}(\phi(p),\phi(p))$$

in some and hence any chart (U,ϕ) around p.

In Section 2, we shall show that the Riemannian distance function of a metric g gives an upper bound for the distance functions satisfying the condition R^∞ associated with g, see Theorem 2.1.

In Section 3, we consider the function

$$L\colon TM\to M$$

defined in the abstract. In Theorem 3.1, we prove that on a complete surface (M,g) with small curvature, small curvature change and injectivity radius $+\infty$ there is a unique vector v, giving a minimum of L on TM. In other words there is a unique geodesic, γ_v which approximates the points $p_1,\dots,p_m$ in the best possible way. Notice that, in case $M=\mathbb{R}$ with the standard metric we are considering the usual linear regression problem, where we find the line in the plane approximating data $(t_1,p_1),\dots,(t_m,p_m)$ in the best possible way. Here L is the sum of least squares.

2. Maximality of the Riemannian distance function

Theorem 2.1. *If $d: M\times M\to\mathbb{R}$ satisfies the condition R^∞ then there exists an open neighbourhood Ω of Δ in $M\times M$, such that*

$$d\le d_g$$

on Ω.

Proof. g has a Riemannian distance function

$$d_g\colon M\times M\to\mathbb{R}.$$

Here d_g^2 is smooth on an open neighbourhood of the diagonal in $M \times M$. In fact, there exists an open neighbourhood Ω of Δ in $M \times M$, such that every $(p, q) \in \Omega$ is contained in a

$$\mathcal{C} \times \mathcal{C},$$

where $\mathcal{C}$ is a convex open set such that

$$d_g(p, q)^2 = \langle \sigma'_{pq}(0), \sigma'_{pq}(0) \rangle,$$

where σ_{pq} is the unique geodesic in $\mathcal{C}$ joining $p = \sigma_{pq}(0) = p$ and $\sigma_{pq}(1) = q$.

Letting $d = d_\phi$, we know that

$$d_\phi(x, \gamma(t)) = t^2 \langle \gamma'(0), \gamma'(0) \rangle + t^3 h(t)$$

for a smooth function h. Now let γ denote a geodesic for the Riemannian metric g. The triangle inequality becomes

$$d(x, \gamma(t+s)) \leq d(x, \gamma(t)) + d(\gamma(t), \gamma(t+s)) + 2\sqrt{d(x, \gamma(t))}\sqrt{d(\gamma(t), \gamma(t+s))}.$$

We shall use this inequality to derive a differential inequality for h. Now

$$d(x, \gamma(t+s)) = (t+s)^2 \langle \gamma'(0), \gamma'(0) \rangle + (t+s)^3 h(t+s)$$

and

$$d(\gamma(t), \gamma(t+s)) = s^2 \langle \gamma'(t), \gamma'(t) \rangle + s^3 h_t(s).$$

Using the standard trick from singularity theory, we write

$$h(t) - h(t+s) = sk(t, s),$$

where k is smooth and

$$k(t, 0) = -h'(t).$$

The triangle inequality above becomes for $s > 0$

$$\begin{aligned} &s(s\langle \gamma'(0), \gamma'(0) \rangle + 2t\langle \gamma'(0), \gamma'(0) \rangle + 3t^2 h(t+s) + 3tsh(t+s) + s^2 h(t+s)) \\ &\quad \leq s(t^3 k(t, s) + s\langle \gamma'(t), \gamma'(t) \rangle + s^2 h_t(s) + 2t\sqrt{|\gamma'(0)|^2 + th(t)}\sqrt{|\gamma'(0)|^2 + sh_t(s)}). \end{aligned}$$

Cancelling the factor s on each side and letting $s \to 0$, we find for $t \geq 0$

$$2t\langle \gamma'(0), \gamma'(0) \rangle + 3t^2 h(t) \leq -t^3 h'(t) + 2t\sqrt{|\gamma'(0)|^2 + th(t)}|\gamma'(0)|.$$

So for $t > 0$

$$h'(t) \leq \frac{1}{t^2}(2\sqrt{|\gamma'(0)|^2 + th(t)}|\gamma'(0)| - 2|\gamma'(0)|^2 - 3th(t)) = \frac{1}{t^2} H(th(t)).$$

Here

$$H(x) = 2\sqrt{|\gamma'(0)|^2 + x}|\gamma'(0)| - 2|\gamma'(0)|^2 - 3x.$$

Notice that

$$H'(x) = \frac{|\gamma'(0)|}{\sqrt{|\gamma'(0)|^2 + x} - 3} < 0$$

for $x \geq 0$. Since $H(0) = 0$, it follows that

$$H(x) < 0, \quad x > 0.$$

Assume for contradiction that there exists $t_* > 0$, such that $h(t_*) = x_0 > 0$. Then

$$h'(t_*) \leq H(t_* h(t_*)) < 0.$$

So

$$h(t) > x_0, \quad t \in]t_* - \delta, t_*[.$$

If there exists $t \in]0, t_*[$, such that $h(t) \leq x_0$ then there exists $\hat{t} \geq 0$, such that

$$h(\hat{t}) = x_0 > 0, \qquad h(t) \geq x_0, \quad t \in]\hat{t}, t_*[.$$

But, then

$$h'(\hat{t}) < 0,$$

which is impossible. Hence

$$h(t) \geq x_0, \quad t \in [0, t_*].$$

In particular,

$$h(0) \geq x_0 > 0$$

contradicting the fact that

$$h(0) = 0$$

according to smoothness of h and the differential inequality for h. Consequently, $h(t) \leq 0$, $t > 0$. Hence the Theorem. □

Example 2.2. We shall now show that the inequality in Theorem 2.1 can be sharp. Consider then

$$S^n \subset \mathbb{R}^{n+1}$$

with the standard Riemannian metric tensor g and the corresponding distance function d_g. The Euclidean norm in $\mathbb{R}^{n+1}$ is denoted by $\| \|_2$. Then

$$d(x, y) = \|x - y\|_2$$

defines a distance function

$$d: S^n \times S^n \to \mathbb{R}$$

satisfying the condition R^{∞}. Let $x, v \in S^n$, $x \perp v$.

$$\gamma(t) = x\cos t + v\sin t, \quad t \in \mathbb{R}$$

is then a geodesic and

$$\frac{\partial^2}{\partial t^2} d^2(x, \gamma(t))_{|t=0} = 2.$$

So d induces the standard Riemannian metric tensor g. But

$$d(x, y) < d_g(x, y), \quad x \neq y, \ \ x, y \in S^n.$$

Example 2.3. Define smooth arc length parameterized curves

$$\alpha_i : I_i \to \mathbb{H},$$

where $\mathbb{H}$ is a Hilbert space and I_i is an open interval in $\mathbb{R}$. Define a distance function on

$$U = I_1 \times \cdots \times I_n$$

by

$$d^2(x, y) = \|\alpha_1(x_1) - \alpha_1(y_1)\|^2 + \cdots + \|\alpha_n(x_n) - \alpha_n(y_n)\|^2.$$

Then d^2 is smooth and induces a flat metric. Also d satisfies the condition R^{∞}.

3. Distance functions and statistics

In this section, let (M, g) denote a complete, smooth Riemannian surface. Assume we are given mutually distinct points

$$p_1, \ldots, p_m \in M$$

and real numbers

$$0 < t_1 < \cdots < t_m, \qquad T = (t_1, \ldots, t_m).$$

Define a non-negative function

$$L : TM \to \mathbb{R}$$

on the tangent bundle TM of M by

$$L(v) = \sum_{i=1}^{m} d(p_i, \gamma_v(t_i))^2, \quad v \in TM.$$

Given $v \in TM$, define

$$K_v = \frac{\langle \nabla_v R(v_1, v_2)v_1, v_2\rangle}{b(v_1, v_2)}, \quad v_i \in T_{\pi(v)}M,$$

$$b(v_1, v_2) = \langle v_1, v_2\rangle\langle v_2, v_2\rangle - \langle v_1, v_2\rangle^2 \neq 0,$$

where R is the curvature tensor of M. Assume the injectivity radius $\iota(M) = +\infty$.

Theorem 3.1. *L has a minimum v_0 on TM. There exist positive functions*

$$Q = Q(T, \{d(p_i, p_j)\}_{i,j}) > 0, \qquad Q_1 = Q_1(T, \{d(p_i, p_j)\}_{i,j}) > 0$$

such that if

$$|K| \leq Q^2, \qquad |K_v| \leq Q_1 \;\; \forall_v \textit{ with } \|v\| = 1,$$

then v_0 is unique.

Proof. *Existence*. Define

$$K_2 = \max_{i\geq 2}(d(p_i, p_1)).$$

Given $v \in TM$, let $p = \pi(v)$. For $d(p, p_1) \leq K_2$ and $\|v\| \leq K_3$ for some $K_3 > 0$, we find

$$\sum_{i=1}^{m} d(p_i, \gamma_v(t_i))^2 \leq \sum_{i=1}^{m}(d(p_i, p_1) + d(p_1, p) + d(p, \gamma_v(t_i)))^2 \leq \sum_{i=1}^{m}(K_2 + K_2 + t_i K_3)^2 \triangleq K_1.$$

There exists $R > K_2$ such that

$$\sum_{i=1}^{m} d(p_i, \gamma_v(t_i))^2 \geq \sum_{i=1}^{m}(d(p_1, p) - t_i\|v\| - d(p_i, p_1))^2 > K_1$$

for $d(p_1, p) > R$. Estimate

$$d(p_i, \gamma_v(t_i)) \geq |d(\gamma_v(t_1), \gamma_v(t_i)) - d(p_1, \gamma_v(t_1)) - d(p_i, p_1)|.$$

If $d(p_1, \gamma_v(t_1)) \leq \sqrt{K_1}$ and $\|v\| \geq K_4$, then

$$\sum_{i=1}^{m} d(p_i, \gamma_v(t_i))^2 \geq \sum_{i=1}^{m}(K_4|t_i - t_1| - \sqrt{K_1} - K_2)^2 > K_1.$$

It remains to consider $\|v\| \in [K_3, K_4]$. For $d(p_1, p) > K_5 > R$, we find

$$d(p_1, \gamma_v(t_1)) \geq d(p_1, p) - d(\gamma_v(t_1), p) = d(p_1, p) - \|v\|t_1 > K_5 - K_4 t_1 > \sqrt{K_1}.$$

So L assumes a minimum on the compact set

$$d(p_1, p) \leq K_5, \qquad \|v\| \leq K_4.$$

This proves existence of v_0.

Uniqueness. Define the two Jacobi map

$$F_2(\sigma_s, \sigma_{st}, \sigma_t) = (R(\sigma_t, \sigma_s)\sigma_s)_t + R(\sigma_t, \sigma_s)\sigma_{st} + (R(\sigma_t, \sigma_s)\sigma_t)_s - R(\sigma_{tt}, \sigma_s)\sigma_s$$

for a two parameter geodesic variation σ of a unit speed geodesic $s \mapsto \sigma(s, 0)$. It gives rise to the two Jacobi equation

$$Y_2'' = F_2(\sigma_s, Y_1', Y_1) + R(Y_2, \sigma_s)\sigma_s, \qquad Y_1 = \sigma_t, \qquad Y_2 = \sigma_{tt},$$

cf. [2]. This is the differential equation for the transverse acceleration vector field of a geodesic variation. Now let $E_1 = \sigma_s$, E_2 denotes an orthonormal basis along $s \mapsto \sigma(s, 0)$. Write

$$\sigma_t = y_1 E_1 + y_2 E_2, \qquad \sigma_{st} = y_1' E_1 + y_2' E_2, \qquad \sigma_{tt} = z_1 E_1 + z_2 E_2.$$

Let

$$K_1 = \langle R_{E_1}(E_2, E_1)E_2, E_1\rangle, \qquad K_2 = \langle R_{E_2}(E_2, E_1)E_2, E_1\rangle.$$

The above second-order differential equation for Y_2 becomes

$$z_1'' = 4y_2 y_2' K + y_2^2 K_1, \qquad z_2'' + Kz_2 = -2y_1 y_2 K_1 - y_2^2 K_2 - 4y_2 y_1' K = y$$

on our surface (M, g).

We shall now prove that the Hessian of L is positive definite near the $p_i's$. To this end consider a non-constant geodesic

$$c: I \to TM$$

with

$$c(0) = v, \qquad c'(0) \neq 0.$$

The base curve

$$d = \pi \circ c$$

is denoted d.

Define a geodesic variation $G = G^i$ with

$$s \mapsto G(s, t)$$

a geodesic from

$$G(0, t) = p_i \text{ to } \exp(t_i c(t)) = G(1, t).$$

Let

$$L_i(v) = d(p_i, \exp(t_i v))^2.$$

Then

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(L_i \circ c)_{t=0} = \frac{\mathrm{d}}{\mathrm{d}s}(\langle G_{tt}, G_s\rangle)_{s=0} - 2\langle R(G_t, G_s)G_t, G_s\rangle + 2\langle G_{ts}, G_{ts}\rangle_{t=0}.$$

Notice that

$$G_t(0, 0) = 0, \qquad G_{tt}(0, 0) = 0.$$

We shall need a second geodesic variation defined by

$$y(s,t)=\exp(sc(t)).$$

Notice here that

$$y_{tt}(0)=(\pi\circ c)''(0).$$

The strategy of the proof is to give an upper estimate for

$$\|y_{tt}(t_i,0)\|=\|G_{tt}(1,0)\|$$

and a lower estimate for

$$\|y_t(t_i,0)\|=\|G_t(1,0)\|.$$

These estimates in turn will give an upper bound for

$$\sum_{i=1}^{m}\left|\frac{\mathrm{d}}{\mathrm{d}s}\langle G_{tt}^i,G_s^i\rangle(0,0)\right|$$

and a lower bound for

$$\sum_{i=1}^{m}\langle G_{ts}^i,G_{ts}^i\rangle(0,0).$$

Thus showing that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(L\circ c)(0)>0.$$

Initially we focus on estimates for the Jacobi field

$$y_t(s,0)=y_1\left(\frac{y_s}{|y_s|}\right)+y_2N,$$

where N is a unit parallel vector field orthogonal to y_s. Here we assume that $|y_s|\neq 0$ dealing with the case $|y_s|=0$ later. Let

$$a=y_1'(0),\qquad b=y_1(0),\qquad Qc=y_2'(0),\qquad d=y_2(0).$$

Let y_2^Q and y_2^{-Q} denote solutions to

$$y_2''-Q^2y_2=0$$

and

$$y_2''+Q^2y_2=0,$$

respectively, with

$$(y_2^Q)'(0)=(y_2^{-Q})'(0)=Qc,\qquad (y_2^Q)(0)=(y_2^{-Q})(0)=d.$$

Consider first the case $d > 0$. We have

$$y_2^{-Q}(t) \leq y_2(t) \leq y_2^{Q}(t)$$

as long as $y_2^{-Q}(t) > 0$. So

$$c \sin(Qt_i) + d \cos(Qt_i) \leq y_2(t_i) \leq c \sinh(Qt_i) + d \cosh(Qt_i).$$

We shall now derive a lower bound for

$$\sum_{i=1}^{m} |Y(t_i)|.$$

First of all

$$|y_1(t_i) - y_1(t_j)| = |a||t_i - t_j| \leq |y_1(t_i)| + |y_1(t_j)|.$$

Hence

$$|a| \sum_{i,j=1}^{m} |t_j - t_i| \leq 2m \sum_{i=1}^{m} |y_1(t_i)|.$$

Furthermore,

$$m|b| \leq \sum_{i=1}^{m} |y_1(t_i)| + |a|t_i \leq \left(1 + \sum_{i=1}^{m} t_i \frac{2m}{\sum_{i,j=1}^{m} |t_j - t_i|}\right) \sum_{i=1}^{m} |y_1(t_i)|.$$

Now

$$d \cos Qt_i \leq y_2(t_i) - c \sin Qt_i.$$

Also

$$-c \sinh Qt_i \leq -y_2(t_i) + d \cosh Qt_i \leq -y_2(t_i) + \frac{\cosh Qt_i}{\cos Qt_j}(y_2(t_j) - c \sin Qt_j).$$

Hence

$$c \left(\frac{\cosh Qt_i}{\cos Qt_j} \sin Qt_j - \sinh Qt_i \right) \leq y_2(t_j) \frac{\cosh Qt_i}{\cos Qt_j} - y_2(t_i). \tag{3.1}$$

For $c > 0$, we have

$$d \cos Qt_i \leq y_2(t_i) - d \cos Qt_i \leq \sum_{i=1}^{m} |y_2(t_i)|.$$

For $c < 0$, we get with $i > j$ from (3.1) for small Q

$$-c \leq \frac{1}{\sinh Qt_i - (\cosh Qt_i / \cos Qt_j) \sin Qt_j} y_2(t_j) \frac{\cosh Qt_i}{\cos Qt_j}.$$

Here we assume that Q is so small such that

$$\sinh Qt_i - \frac{\cosh Qt_i}{\cos Qt_j} \sin Qt_j > 0.$$

All in all, we have an estimate

$$\sum_{i=1}^{m} |Y(t_i)| \geq \tilde{K}(|a| + |b| + |c| + |d|).$$

It may happen that $y_2(t_*) < 0$ for some $t_* > 0$. If there are at least two $t_i's$ for which

$$y_2(t_i) < 0$$

we can argue as follows.

By y^Q we denote the solution to

$$y_2'' - Q^2 y_2 = 0$$

with $y^Q(0) = d$, $y^Q(t_*) = 0$, t_* being the first zero for y_2. So

$$y^Q(t) = c\cosh(Qt) + d\sinh(Qt).$$

Assume for contradiction that

$$(y^Q)'(0) > y_2'(0).$$

Then

$$y_2'' = -Ky_2 < Qy_2 < Qy^Q = (y^Q)''.$$

So

$$y^Q(t_*) > y_2(t_*) = 0.$$

A contradiction and $(y^Q)'(0) \leq y_2'(0)$. Now we deduce that

$$y^Q(t) \leq y_2(t), \quad t \in]0, t_*[.$$

Hence

$$(y^Q)'(t_*) > y_2'(t_*) = \tilde{c}.$$

Argue as before to obtain

$$|\tilde{c}| \leq \sum_{i=1}^{m} |y_2(t_i)|.$$

Now compute

$$(y^Q)'(t_*) = -Q\sqrt{((y^Q)'(0))^2 - d^2}.$$

Hence

$$\tilde{K}_3|c|,\, \tilde{K}_3 d \le K_3|(y^Q)'(t_*)| \le K_3|\tilde{c}| \le \sum_{i=1}^{m}|y_2(t_i)|$$

for small Q. We have

$$x_1^i(s) = a_1^i s, \qquad |x_2^i(s)| \le |c_1^i| \sinh(Qs), \qquad (x_2^i)'(0) = Qc_1^i.$$

Hence

$$\tilde{K}(|a| + |b| + |c| + |d|) \le \sum_{i=1}^{m}|Y(t_i)| = \sum_{i=1}^{m}\left|\frac{\partial G^i}{\partial t}(1,0)\right| = \sum_{i=1}^{m}|x_1^i(f_i)| + |x_2^i(f_i)|$$
$$\le \sum_{i=1}^{m}|a_1^i| f_i + |c_1^i| \sinh(Qf_i)$$

$f_i = d(p_i, \gamma_v(t_i)) < f_0$. We obtain an estimate

$$\sum_{i=1}^{m}\langle G_{ts}^i, G_{ts}^i\rangle(0,0) \ge \sum_{i=1}^{m}\frac{1}{f_i}\hat{K}(|a| + |b| + |c| + |d|)^2.$$

To derive an upper bound for

$$\sum_{i=1}^{m}\left|\frac{\mathrm{d}}{\mathrm{d}s}\langle G_{tt}^i, G_s^i\rangle\right|,$$

we observe that

$$z_2(t) = a(t)\int_0^t \frac{B(s)}{a(s)^2}\,\mathrm{d}s + \hat{z}_2(t), \qquad B(t) = \int_0^t y(s)a(s)\,\mathrm{d}s,$$

where a is the solution to

$$a'' + Ka = 0, \qquad a'(0) = 1, \qquad a(0) = 0$$

and $\hat{z}_2$ is the solution to

$$\hat{z}_2'' + K\hat{z}_2 = 0, \qquad \hat{z}_2(0) = z_2(0), \qquad \hat{z}_2'(0) = z_2'(0).$$

Taking Q small we have $a(t_i) > 0$.

We shall need to examine the differential equations for c, the geodesic in TM. So, write locally

$$c^\phi = (d, E)\colon I \to \phi(U) \times \mathbb{R}^n.$$

Here (U, ϕ) is a chart on M such that

$$g_{ij}(p) = \delta_i^j, \qquad \pi(v) = p, \qquad \Gamma_{ij}^k(p) = 0.$$

By $G_{\alpha\beta}$ we denote the coordinates of the Sasaki metric, see [1], in the tangent bundle chart associated with ϕ and we use the terminology

$$\alpha = (i, I), \qquad \beta = (j, J), \quad i, j, I, J \in \{1, \dots, n\}.$$

We have

$$G_{ij}(p) = g_{ij}(p) + g_{ab}\Gamma^a_{dj}(p)v^c\Gamma^b_{dj}(p)v^d = g_{ij}(p), \qquad G_{iJ}(p) = [ki, J]_p v^k = 0,$$
$$G_{IJ}(p) = g_{IJ}(p).$$

We have

$$\frac{\mathrm{d}^2 E_K}{\mathrm{d}t^2} + \Gamma^{K,N}_{\alpha\beta} c'_\alpha c'_\beta = 0, \quad N = TM.$$

Direct computation using the formulas for $G_{\alpha\beta}$ above yields

$$\Gamma^{M,N}_{ij}(p) = \frac{1}{2}\left(\frac{\partial}{\partial x_j}[ki, M] + \frac{\partial}{\partial x_i}[kj, M]\right)_p v^k, \quad \Gamma^{M,N}_{iJ}(p) = 0, \quad \Gamma^{M,N}_{I,J}(p) = 0.$$

It follows that

$$c_{tt} = \left(\frac{\mathrm{d}^2 E^K}{\mathrm{d}t^2} + \frac{\partial}{\partial x_l}(\Gamma^K_{ij}(p) d'_l d'_i E_j\right)\partial_K = 0.$$

So there exists two parallel vector fields E_1 and E_2 along d, such that

$$c(t) = tE_1(t) + E_2(t).$$

The differential equations for d are

$$\frac{\mathrm{d}^2 d_k}{\mathrm{d}t^2} + \Gamma^{k,N}_{\alpha\beta} c'_\alpha c'_\beta = 0.$$

Now

$$\Gamma^{k,N}_{ij}(p) = \Gamma^{k,M}_{ij}(p) = 0, \qquad \Gamma^k_{IJ}(p) = 0.$$

So

$$\frac{\mathrm{d}^2 d_k}{\mathrm{d}t^2} = -2\Gamma^{k,N}_{iJ} d'_i E'_j,$$

where

$$\Gamma^{k,N}_{iJ} = R^J_{aki} v^a.$$

We consequently have a bound

$$|z_1(0)|, |z_2(0)| \le |d''(0)| \le LQ^2(|a| + |b| + |c| + |d|)^2|v|$$

for the acceleration of the base curve $d = \pi \circ c$ of c. Furthermore,

$$|z'_1(0)|, |z'_2(0)| \le |y_{tts}(0, 0)| = |R(v, d')d'| \le \tilde{L}Q^2(|a| + |b| + |c| + |d|)^2|v|.$$

We have estimates

$$|y_1(s)| \leq |a|s + |b|, \qquad |y_2(s)| \leq 2|c|s + 2|d|$$

and

$$b_1 s \leq a(s) \leq b_2 s, \quad 0 < b_1 < 1 < b_2.$$

We have for some $\tilde{L} > 0$ such that

$$|B(t)| \leq \int_0^t (|a| + |b| + |c| + |d|)^2 \tilde{L}(Q^2 + Q_1) s \, \mathrm{d}s$$
$$\leq \tilde{L}(|a| + |b| + |c| + |d|)^2 (Q^2 + Q_1) \frac{1}{2} t^2.$$

Hence

$$|z_2(t_i)| \leq L_1(|a| + |b| + |c| + |d|)^2 (Q^2 + Q + Q_1).$$

Similarly, integrating twice

$$|z_1(t_i)| \leq L_2(|a| + |b| + |c| + |d|)^2 (Q^2 + Q + Q_1).$$

Now we can conclude the proof by observing that for $s = f_i$ we have

$$v_1(s) = v_1'(0)s + \int_0^s \int_0^u (4x_2 x_2' K + x_2^2 K_1) \, \mathrm{d}\tau \, \mathrm{d}u = \left\langle \frac{G_{tt}, G_s}{|G_s|} \right\rangle.$$

So

$$f_i |v_1'(0)| \leq \left| \left\langle \frac{G_{tt}, G_s}{|G_s|} \right\rangle - \int_0^f \int_0^u (4x_2 x_2' K + x_2^2 K_1) \, \mathrm{d}\tau \, \mathrm{d}u \right|$$
$$\leq \sqrt{z_1(t_i)^2 + z_2(t_i)^2} + \hat{L}(Q^2 + Q_1)|c_1|^2.$$

Hence

$$\left| \frac{\mathrm{d}}{\mathrm{d}s} \left\langle \frac{G_{tt}, G_s}{|G_s|} \right\rangle \right| \leq \sum_{i=1}^m \frac{1}{f_i} L(|a| + |b| + |c| + |d|)^2 (Q + Q^2 + Q_1).$$

Comparing the summands of the Hessian of L we see that this Hessian is positive definite near the $p_i' s$. Uniqueness of v_0 follows, thereby proving the theorem. □

Example 3.2. Suppose (M, g) is a complete surface with a closed geodesic γ of period $T > 0$ with

$$\gamma(t) = \gamma(t + T), \quad t \in \mathbb{R}.$$

Let $3\tau \in]0, T[$ and set

$$p_1 = \gamma(\tau), \qquad p_2 = \gamma(2\tau), \qquad p_3 = \gamma(3\tau).$$

Define

$$\beta(t) = \gamma\left(-T + \frac{T+\tau}{\tau}t\right)$$

with

$$\beta'(0) = \frac{T+\tau}{\tau}\gamma'(0)$$

and

$$\beta(\tau) = p_1, \qquad \beta(2\tau) = p_2, \qquad \beta(3\tau) = p_3.$$

Then

$$L(v) = \sum_{i=1}^{3} d(p_i, \gamma_v(t_i))^2$$

has at least two distinct minima $\gamma'(0)$ and $((T+\tau)/\tau)\gamma'(0)$. Here $\iota(M) < +\infty$.

References

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